Functional Equations

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1 Introduction

Functional equations are sort of a strange area of math olympiads, as most of the problems will be fairly contrived and not super natural. In that sense they are like an abstract puzzle, where the best way to learn and get better is through experience. A classic functional equation will look like:

Find all functions $f : A \to B$ (some sets A, B) satisfying
[insert equation(s) here]
for all [insert variables here]$\in A$.

Now, there are plenty of variants, including

1. the equality is replaced by an inequality;
2. you are asked to prove a certain property of such functions instead of finding all possibilities;
3. you are asked to construct a function satisfying the equation(s).

Each variant has it's own little nuances, but they all share one thing: to solve the problem you have to play around with things.

The classical functional is by far the most common of the variants. One of the first steps in such a problem is to attempt to determine what the solutions will be. The most common solutions will be

1. $f(x) = c$ a constant;
2. $f(x) = x$, or more generally a subset of the linear equations $f(x) = ax + b$;
3. $f(x) = x^2$;
4. $f(x) = \frac{1}{x}$ or $f(x) = \frac{1}{1-x}$.

Of course there are many more unusual and exotic possibilities, but more often than not they won't occur.

Determining the set of solutions is typically worth a point, and it should also motivate your attack on the problem. For example, if one solution is not injective, then you will have a hard time proving that $f(x)$ must be injective. Alternatively, assume that you decide that the set of solutions is $f(x) = 0, x$. It would be impossible to prove that $f(1) = 1$, since this is not necessarily true.

What will most likely happen here is you will divide into 2 cases: one being $f(1) = 0$ (from which you hope to conclude that $f(x) = 0$ for all x), and the other being $f(1) \neq 0$ (from which you look for $f(1) = 1$, and more generally $f(x) = x$).

It is important to check that your proposed solutions work, as you can lose points if you do not. If you go with the line "and we have checked that the proposed solutions all satisfy the given equation", make sure that there is a few lines of algebra nearby which clearly demonstrate that you have checked the solutions, especially if they are exotic. I once lost 2 points for committing this sin: one for not checking my valid solutions, and one for including an extraneous solution (which I would have discarded if I had checked).

2 Strategies

Here is a list of general strategies:

1. Make terms on each side of the equation equal, and cancel them.
2. Take specific choices of the $x, y, \ldots$ to make nice equations.
3. Try to determine $f(0)$,$f(1)$, etc.
4. Make all variables equal, or similar variants.
5. Try to prove injectivity or surjectivity
6. If you know that 0 is in the image of f, take x to be such that $f(x) = 0$ (and variants of this)
7. Replace the variables (x, y) by variants like (y, x), $(f(x), y)$, $(x + y, y)$, etc.

Another specific tool is Cauchy's functional equation: if $f : \mathbb{R} \to \mathbb{R}$ satisfies $f(x + y) = f(x) + f(y)$ for all x, y, then $f(x) = f(1)x$ for all rational numbers x. Therefore if the domain of f is $\mathbb{Q}$, we can conclude that $f(x) = kx$ for some fixed k.

Proof. Taking $x = y$ gives $f(2x) = 2x$, $(x, y) = (2x, x)$ gives $f(3x) = 3f(x)$, and we can induct to show that $f(nx) = nf(x)$ for all positive integers n. Taking $x = y = 0$ gives $f(0) = 0$, and $(x, y) = (x, -x)$ gives $f(-x) = -f(x)$, whence $f(nx) = nf(x)$ for all integers n. If $\frac{a}{b}$ is a rational number, we thus see that $bf(a/b) = f(a) = af(1)$, whence $f(a/b) = (a/b)f(1)$ as required. □

Other notes:

1. If you are asked to construct a function, it is often difficult to give a closed form expression. You will likely have to construct it with some sort of induction procedure.
2. For inequalities, there is often much less that you can do. Contradictions normally come from obtaining the expression $A < A$, and equalities normally come from a sequence of inequalities like
$$\text{expression}_1 \leq \text{expression}_2 \leq \text{expression}_1,$$
from which we conclude $\text{expression}_1 = \text{expression}_2$

3. Injectivity/surjectivity is difficult to prove unless you have a variable apart from the inside of a function call. For example in $f(x+y) = f(x^2 - f(y)) + f(y-x)$ it would be difficult to prove either.

4. If you obtain the equation $f(f(x)) = x$, then note that f is now automatically injective and surjective.

5. Beware the "pointwise trap"; for example, you may prove that $f(x)^2 = x^2$, hence $f(x) = \pm x$. This is only necessarily true pointwise! To conclude that the only solutions are $f(x) = x$ for all x and $f(x) = -x$ for all x, you will need to use other equations again (otherwise we could have something weird like $f(x) = x$ if x is rational and $f(x) = -x$ if x is irrational; this needs to be ruled out).

3 Example

Example 1 (From David Yang via Evan Chen's online notes found at `http://web.evanchen.cc/handouts/FuncEq-Intro/FuncEq-Intro.pdf`).
Solve over $\mathbb{R}$*:*

$$f(x^2 + y) = f(x^{27} + 2y) + f(x^4).$$

Solution. The terms are pretty nasty, so let's try to cancel them. If we set $x^2 + y = x^{27} + 2y$, then this is equivalent to $y = x^2 - x^{27}$. With this choice of y (given any x), these terms cancel and we are left with $f(x^4) = 0$. Thus $f(x) = 0$ for all $x \geq 0$. To finish off, we now have $f(x^2+y) = f(x^{27}+2y)$. We can take $y = 0$ to see $f(x^{27}) = f(x^2) = 0$, so $f(x) = 0$ for all x (since 27 was odd). □

Example 2 (CMO 2008 P2)**.** *Determine all functions* $f : \mathbb{Q} \to \mathbb{Q}$ *which satisfy*

$$f(2f(x) + f(y)) = 2x + y$$

for all $x, y \in \mathbb{Q}$*.*

Solution. Clearly f is surjective, and f is also injective (indeed, if $f(a) = f(b)$ then take $y = a = b$ and x to be anything). Since f is surjective, take $x = y = a$ where $f(a) = 0$. Then $f(0) = 3a$. Taking $x = y = 0$ we then get $f(9a) = 0 = f(a)$. Since f is injective, this implies $9a = a$, i.e. $a = 0$. Therefore we have $f(0) = 0$.

With this in hand the equation falls apart. Take $y = 0$ to get $f(2f(x)) = 2x$, and take $x = 0$ to get $f(f(y)) = y$; replacing x by $f(x)$ in $f(2f(x)) = 2x$ now gives $f(2x) = 2f(x)$. Replace (x, y) by $f(x), f(y)$ in the original equation, and these equations give us

$$f(2x) + f(y) = 2f(x) + f(y) = f(2f(f(x)) + f(f(y))) = f(2x + y).$$

Replacing $2x$ by x, we see that $f(x + y) = f(x) + f(y)$ for all x, y. This is Cauchy's functional equation, and as our inputs and outputs are rational, the only solutions are $f(x) = kx$ for $k \in \mathbb{Q}$. Plugging this back into the original equation gives us $k = \pm 1$, and $f(x) = x, -x$ are the two solutions. □

4 Problems

I will adopt the David Arthur style of ordering problems into 3 sections: A,B,C. A-level problems should be approximately CMO level, B level problems would be easy-medium IMO problems, and C level would be medium-hard IMO or beyond problems. This ordering is of course somewhat subjective, so don't be surprised if you find some problems to be out of place.

A1 Find all functions f from the plane to $\mathbb{R}$ such that for every square $ABCD$ in the plane, we have $f(A) + f(B) + f(C) + f(D) = 0$.

A2 Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function such that

$$f(x, y) + f(y, z) + f(z, x) = 0$$

for all reals x, y, z. Prove that there exists a function $g : \mathbb{R} \to \mathbb{R}$ such that $f(x, y) = g(x) - g(y)$ for all real numbers x, y.

A3 Let S be a set of functions from $\mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ which satisfies:

(a) $f_1(x) = e^x - 1$ and $f_2(x) = \ln(x + 1)$ are in S;

(b) If $f(x), g(x) \in S$, then the functions $f(x) + g(x)$ and $f(g(x))$ are in S;

(c) If $f(x), g(x) \in S$, and $f(x) \geq g(x)$ for all $x \geq 0$, then the function $f(x) - g(x) \in S$.

Prove that if $f(x), g(x) \in S$, then $f(x)g(x) \in S$.

A4 Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x^2 + f(y)) = f(xy)$$

for all real numbers x, y.

A5 Let $f, g : \mathbb{R} \to \mathbb{R}$ be functions that satisfy

$$f(x + g(y)) = 2x + y$$

for all reals x, y. Determine $g(x + f(y))$ (in terms of x, y).

A6 Suppose $f : \mathbb{R} \to \mathbb{R}$ is a function satisfying

$$f(xy) + f(y - x) \geq f(y + x)$$

for all real numbers x, y.

(a) Give a non-constant polynomial satisfying the inequality

(b) Prove that $f(x) \geq 0$ for all x.

A7 Do there exist integers m, n and a function $f : \mathbb{R} \to \mathbb{R}$ which satisfy the following:

(a) $f(f(x)) = 2f(x) - x - 2$ for all $x \in \mathbb{R}$;

(b) $m \leq n$ and $f(m) = n$.

See also question C8.

A8 Prove there do not exist any functions $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ such that for all $x \in \mathbb{Z}^+$ we have

$$f(2f(x)) = x + 1998.$$

A9 Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x+y) + y \le f(f(f(x)))$$

for all reals x, y

B1 Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x^2 + yf(x)) = xf(x+y)$.

B2 Let $f : \mathbb{Z} \to \mathbb{Z}^+$ be a function. Suppose that for any two integers m, n, we have $f(m-n) \mid f(m) - f(n)$. Prove that if m, n are integers with $f(m) \le f(n)$, then $f(m) \mid f(n)$.

B3 Determine all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that for all $x, y \in \mathbb{Z}$ we have

$$f(x - f(y)) = f(f(x)) - f(y) - 1.$$

B4 Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$ we have

$$f(yf(x+y) + f(x)) = 4x + 2yf(x+y).$$

B5 Find all surjective functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that for all $x > 0$ we have

$$2xf(f(x)) = f(x)(x + f(f(x))).$$

B6 Determine all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all reals x, y we have

$$f(y^2 + 2xf(y) + f(x)^2) = (y + f(x))(x + f(y))$$

B7 Find all pairs of polynomials $p(x), q(x)$ with real coefficients such that

$$p(x)q(x+1) - p(x+1)q(x) = 1.$$

B8 Does there exist a function $f : \mathbb{Q} \to \{-1, 1\}$ such that if x, y are distinct rational numbers with $xy = 1$ or $x + y \in \{0, 1\}$, then $f(x)f(y) = -1$?

B9 Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that for all integers a, b, c with $a + b + c = 0$ we have

$$f(a)^2 + f(b)^2 + f(c)^2 = 2f(a)f(b) + 2f(b)f(c) + 2f(c)f(a).$$

C1 Let $K = \{(x, y) : x, y \in \mathbb{Z}\}$ be the set of lattice points in the plane. Does there exist a bijection $f : \mathbb{Z} \to K$ such that for all distinct $a, b, c \in \mathbb{Z}$ with $\gcd(a, b, c) > 1$ we have that $f(a), f(b), f(c)$ are not collinear?

C2 Find all injective functions $f : \mathbb{R}^\times \to \mathbb{R}^\times$ such that

$$f(x+y)(f(x)+f(y)) = f(xy)$$

for all non-zero reals x, y with $x + y \neq 0$.

C3 Let $f : \mathbb{R} \to \mathbb{R}$ be a function which satisfies

$$f(x+y) \leq yf(x) + f(f(x)).$$

for all reals x, y. Prove that $f(x) = 0$ for $x \leq 0$.

C4 Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all real numbers x, y, we have

$$f(f(x)f(y)) + f(x+y) = f(xy).$$

C5 Find all functions $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ which satisfy the equation

$$f^{abc-a}(abc) + f^{abc-b}(abc) + f^{abc-c}(abc) = a + b + c$$

for all $a, b, c \geq 2$. ($f^1(n) := f(n)$ and $f^k(n) := f(f^{k-1}(n))$ for $k \geq 2$).

C6 Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all real numbers x, y, we have

$$f(x + f(x+y)) + f(xy) = x + f(x+y) + yf(x).$$

C7 Let k be a positive integer and $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ a function. We call f $k-$good if we have

$$\gcd(f(m) + n, f(n) + m) \leq k$$

for all distinct positive integers m, n. Prove that there does not exist a $1-$good function, but there does exist a $2-$good function.

C8 Exhibit a function $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$f(f(x)) = 2f(x) - x - 2$$

for all reals x. Note: I don't know the solution or actually how hard it is.

C9 Let n be a given integer. Find all functions $f : \mathbb{Z} \to \mathbb{Z}$, such that for all integers x, y we have

$$f(x + y + f(y)) = f(x) + ny.$$

5 Hints

A1 The only function is $f(P) = 0$ for all P; try to prove this with a nice geometric construction.

A2 Try to show $f(0,0) = 0$ and $f(x,y) = -f(y,x)$.

A3 Can you show $(f(x)+1)(g(x)+1) - 1 = f(x)g(x) + f(x) + g(x) \in S$?

A4 f must be a constant. Try to first show that $f(x) = f(0)$ if $x \geq f(0)$.

A5 Try replacing x with $x - g(y)$.

A6 a) Try to find a quadratic which works. b) Can you cancel terms?

A7 The answer is no. Try $x = m, f(m), f(f(m)), \ldots$ and see if you show $f^k(m) = m$ for some k.

A8 Note that f is injective and $2f(x)$ is even.

A9 Show that $f(f(f(x))) = f(x)$, and show that $f(x) + x$ is constant.

B1 Try to show that $f(0) = 0$, and $f(a) = 0$ with $a \neq 0$ implies that $f(x) = 0$ for all x. Use this to show $f(x) = 0, x$.

B2 Show that $f(n) \mid f(0)$ for all n and that $f(-n) = f(n)$. Now play around with $m, n, m-n, m+n$.

B3 Show that $-1 \in \text{Im}(f)$, and use this to eventually show that $f(-1)+1 = f(x+1) - f(x)$, implying that f is linear.

B4 Try to show that $f(f(x)) = x$, $f(0) = 0$, and $f(1) = 2$.

B5 Prove f is injective, and find an expression for $f^{(n)}(x)$ in terms of $x, f(x)$, and use this to get an inequality. Let $g(x)$ be the inverse of $f(x)$, and look for an equation that it satisfies.

B6 Try to show that there is exactly one value of x such that $f(x) = 0$, as well as showing that $f(0) = 0, 1/2$. Split into cases to try to find the three solutions.

B7 Try replacing x by $x - 1$ and subtracting the equations.

B8 Yes there does; consider the equalities $s(x) = s(x+1)$ and $s(x) = -s(1/x)$; does this remind you of something?

B9 Do not attempt this problem, it is a terrible problem. But seriously, this question is a good test of casework and exotic solutions. There are three families of solutions, with one having $f(x)$ having period 2, and another with $f(x)$ having period 4.

C1 Yes. Try to recursively construct such a function, gradually filling in squares centred at the origin.

C2 Let $a = f(1)$, and rule out possibilities until you get $a = 1$. Now try to show that $f(xy) = f(x)f(y)$ for $x, y \neq 0$ (there will likely be lots of messy algebra in this question).

C3 Try to show: $f(x) \leq 0$ for $x \leq 0$, $f(-\text{big}) = \text{small}$, and $f(f(x)) \geq f(x)$.

C4 Show that $f(0) = 0$ implies $f(x) = 0$. With $f(0) \neq 0$, try setting $x + y = xy$. Try to prove that f is injective, and show how injectivity finishes the problem.

C5 Show that $f^{x^2-x}(x^2) = x$ for $x \geq 2$. Now try to show that $(f^{abc-a}(abc)-a)+(f^{abc-b}(abc)-b) = (f^{abc-ab}(abc) - ab)$, and continue.

C6 Show that $f(0) = 0, 2$. If $f(0) = 2$ show that $f(x) = 2 - x$, otherwise try to show that $-f(x) = f(-x)$, and set up two equations involving $f(x), f(-x), f(-x^2)$.

C7 To disprove the existence of a 1-good function, look modulo 2. For a 2-good function, try to construct it by induction; something should go a little wrong. Try to modify your induction appropriately (there is an alternate explicit construction).

C8 No idea; supposedly Iurie Boreico has proven this on a non-AOPS forum. So if you are a Googling wizard, maybe you can find it!

C9 Define $A = \{y : f(x + y) = f(x) \text{ for all } x\}$, and prove that $A = m\mathbb{Z}$ for some integer m. The cases $n < 0$, $n = 0$, $n > 0$ will be slightly different.